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A Relational Theory of Physical Space

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Abstract

Our problem is to build a relational theory of space, i.e., one according to which space is a sort of network of relations among things. We take the notions of concrete thing and of action of one thing upon another as undefined, or rather as defined in another context. We define the notion of interposition between things in terms of the previous notions. We then define the separation between two things as the set of things interposed between them. The collection of things equipped with the separation function is called the *thing space*—a representation of ordinary space sufficient for philosophical purposes but not for physics. The next step is to define a topology for the thing space: This is done with the help of the separation function. The set of things together with this topology is called the *physical space.* We then define the family of balls lying between any two things and postulate that it satisfies Huntington's axioms for solid geometry. By adding a few more natural assumptions we render physical space a three-dimensional manifold, which is what current physical theories require. We abstain from any metrical considerations, not only because these would require building space-time, but also because our problem was not to describe space but to explain how it comes about. Nevertheless our construction of space involves the notions of event and of event composition, and the latter allows one to define a time order of events, which in turn is required to define the notion of action of one thing upon another. The upshot is a full-fledged relational and objectivistic theory of space based on the assumption that the physical world is constituted by changing things.

1. *Introduction*

The aim of this paper is to sketch a relational theory of physical space. In any relational, as opposed to an absolute theory, space is not assumed to exist by itself, independently of its contents. Instead, a relational theory of space regards the latter as a certain structure of the entire collection of entities of a fundamental kind. These entities are usually taken to be concrete things—such as particles or fields—or events, or both.

The theory to be sketched below takes two notions as undefined: those of thing, or concrete object, and property of a thing. The concepts of state of a thing, and change of state (or event) are also used but are obtained by definition. These four notions are elucidated elsewhere (Bunge, 1974, Bunge, 1977, and Bunge and Sangalli, 1977). In the present paper an extremely brief characterization of these notions will have to suffice.

A fifth notion we need is that of the history of a thing in the state space of the latter. This concept allows one to define the action or effect of one thing upon another, namely, as the change in the history of the patient brought about by the agent. A further defined concept is that of time order, characterized in terms of the concept of event composition: If two events compose in a given order to form a third event then the first precedes the second. The first three sections of the paper are devoted to a presentation of the six above-mentioned key concepts.

In Section 4 we introduce our first and basic, though not primitive, geometric notion, namely, that of betweenness or separation. Roughly, a thing b interposes between things a and c if an action originating in a (or in c) arrives at b before it does at c (or at a). In Section 5 we define the separation between two things as the collection of things interposed between the given things. The separation function allows us in Section 6 to define a topology for the set of things. We postulate that the set of things equipped with that topology represents physical space.

The space of things is assigned further properties in Section 7. Here we postulate that the separation between any two things is a Euclidean ball. We also assume that these Euclidean balls obey Huntington's postulates for ordinary three-dimensional Euclidean geometry. This allows us to prove that the set of things has a covering formed by Euclidean balls and, finally, that the space representing physical space is connected and separable as well as three dimensional. In short, we prove that the space of things is a three-dimensional connected and metrizable manifold.

The upshot of our investigation is that it is possible to define physical space with the help of the general concepts of thing and change. The resulting space is a three-dimensional manifold. This is of course the minimal structure called for by contemporary physics, from quantum mechanics to general relativity.

The present investigation does not extend to space-time but it is generalizable to it. Surely a specification of any metric properties calls for the blending of space with time. But in this paper we are concerned with the gross structure not with the details of space. Ours is not the specific scientific problem of the exact geometry of space(time) but rather the foundational and philosophical problem *What is space?*

2. *Things*

We assume that the world is constituted by things such as elementary particles and fields and the systems composed thereof. Obviously, since we want to build a concept of space in terms of the concept of a thing, we cannot define the latter as whatever exists in space. Things must then be characterized by properties other than spatiotemporal properties.

We submit that the basic property of a thing is that it can associate with other things to form complex things. This association we call *physical sum* and construe as a binary operation $\dot{+}$ in the set T of all things. That is, if x and y are in T, then $x \dot{+} y = z$ is a third thing—one with parts x and y. More explicitly, we define the part-whole relation $\sqsubset$ by

$$\text{If } x, y \in T, \text{ then } x \sqsubset y =_{\text{df}} x \dot{+} y = y$$

The set T of things is assumed to be closed under physical sum, which is taken to be associative, commutative, and idempotent. That is, if x, y, z are in T, then $x \dot{+} (y \dot{+} z) = (x \dot{+} y) \dot{+} z, x \dot{+} y = y \dot{+} x$, and $x \dot{+} x = x$. We also postulate that every nonempty set $S \subseteq T$ of things has a supremum or l.u.b. $[S]$ with respect to the part-whole relation $\sqsubset$. In the case of a finite collection S, $[S]$ is just the physical sum of its members. For example, if $S = \{a, b\}$, then $[S] = a \dot{+} b$. As for the totality T of things, the physical sum of its members, i.e., $[T]$, is called the world or universe. In other words, the world is the aggregation of all things. In short, we assume that $\langle T, \dot{+} \rangle$ is a sup semilattice.

Finally we postulate that every thing is the physical sum or aggregation of a set of basic things. The basics can be particles, fields, or whatever physics may decide. We assume then that (i) there is a nonempty subset B of T—the *basics*—such that, for each x in T, there is a unique subset $B_x \subset B$ such that $x = [B_x]$; and (ii) the basics have no parts: for any $x, y \in B$, if $x \sqsubset y$ then $x = y$. Consequently we can restrict much of our discourse to the basics. In particular the aggregation function will be restricted to 2^B. Moreover by (i) above $[\ ] \mid 2^B : 2^B \to T$ is a bijection. This restriction to basic things has the advantages of economy; in particular we need not refer again to the universe.

3. *States, Events, and Histories*

Every thing has a number of properties in addition to its capacity to aggregate with other things. Whereas some properties (like energy) are frame dependent, others (like composition) are frame invariant. We assume that every thing has a finite (though possibly very large) number of properties with respect to any given reference frame.

Every property of a thing is representable by a (state) function. The collection of all properties of a thing, relative to a given reference frame, can be construed as an n-tuple of functions, with as many components as properties are being represented. (We count every component of a tensor and every component of a complex function as a separate state function.) We call this n-tuple $\mathbb{F} = \langle F_i \mid 1 \leqslant i \leqslant n \rangle$ the *state function* of the thing of interest relative to the given frame of reference.

Every value of $\mathbb{F}$ is a (conceivable) *state* of the thing relative to the given frame. The set of all such states is called the (conceivable) *state space* of the thing. Each kind of thing has its peculiar state space. Now, the components of a state function are subject to restrictions and are interrelated. These restrictions and interrelations are, respectively, the constraints and laws of the thing. As a consequence of such constraints and laws, not every conceivable state is a really possible, or lawful, state. The lawful states of a thing x form a subset $S_{\mathbb{L}}(x)$ of the conceivable state space of x, namely, the *lawful state space* of x. (Because of this restriction, $S_{\mathbb{L}}(x)$ may fail to be a vector space, whence $\mathbb{F}$ is not necessarily a vector.)

Usually the state function $\mathbb{F}$ is a certain function from a certain domain D into $\mathbb{R}^n$. It is convenient and usual to take D to be the state space $S(f)$ of some standard thing or reference frame f, i.e., $\mathbb{F}: S(f) \to \mathbb{R}^n$. In this way every thing state s is the value of $\mathbb{F}$ for some state t of the frame, i.e., $s = \mathbb{F}(t)$ for t in $S(f)$. (We need not go here into the characterization of a reference frame. Suffice it to say that a reference frame is a thing of a special kind, natural or artificial, allowing one to parametrize the states of any things of interest. A possible realization of a reference frame is a clock mounted on a ruler.)

Things are not quiescent but rather restless: They undergo changes, events or processes. An *event* is defined in our system as a change of state of some thing. If a thing x changes from state s to state s' along a curve $g: S_L(x) \to S_L(x)$ in its lawful state space $S_L(x)$, then the net change or event occurring in the thing is represented by the ordered triple $\langle s, s', g\rangle$ with $s' = g(s)$. Every process occurring in x is representable by an arc of curve in the lawful state space of x. Such arcs of curve are called *histories*. More exactly, the full history of a thing x relative to a frame f is

$$h(x) =_{\mathrm{df}} \{\langle t, \mathbb{F}(t)\rangle \mid t \in S(f)\}$$

Let us now characterize the notion of action or effect of one thing upon another. Let x and y be two different things with state functions $\mathbb{F}$ and $\mathbb{G}$ relative to a common reference frame f, and call

$$h(x) = \{\langle t, \mathbb{F}(t)\rangle \mid t \in S(f)\}, \qquad h(y) = \{\langle t, \mathbb{G}(t)\rangle \mid t \in S(f)\}$$

their respective histories. Further, let $\mathbb{H} = g(\mathbb{F}, \mathbb{G}) \neq \mathbb{G}$ be a third state function depending on both F and $\mathbb{G}$, and call

$$h(y \mid x) = \{\langle t, \mathbb{H}(t)\rangle \mid t \in S(f)\}$$

the corresponding history. Then we say that x *acts* upon y, or $x \rhd y$, if the modified trajectory differs from the free one, so that the total history is

$$h(x) \cup h(y \mid x) \neq h(x) \cup h(y)$$

Finally the total action or *effect* of x on y equals the difference between the forced trajectory and the free trajectory of the patient y, i.e.,

$$\epsilon(x, y) =_{\mathrm{df}} h(y \mid x) \cap \overline{h(y)}$$

We shall use the concepts of action ($\rhd$) and of total effect (ϵ) in building our notion of betweenness or interposition.

4. *Time Order*

Consider the net events, or changes of state, occurring in a certain thing x, i.e., disregard the intermediate steps between the initial and the final states. Each such net event is a pair of states $\langle s, s' \rangle \in S_L(x)$, and the totality of such pairs is a set $E_L(x)$, included in $S_L(x) \times S_L(x)$, called the *event space* of x relative to the given frame of reference. And we denote by $E'_L(x)$ the set of *proper* events, i.e., $E_L(x)$ minus the set of identical state transitions $s \mapsto s$. (The inclusion is proper because not all pairs of states are really possible, i.e., lawful: Just think of the conceivable but physically impossible transitions that violate any conservation laws.)

Let s, s', s'', and s''' be four states of x, and let

$$e = \langle s, s' \rangle, \qquad e' = \langle s'', s''' \rangle$$

be two events in x. These events can compose to form a third event in x just in case the terminal state of the first event coincides with the beginning of the second. More precisely, we introduce a partial binary operation $*$ on the proper event space $E'_L(x)$ such that

$$e * e' = \begin{cases} \langle s, s''' \rangle \text{ if } s' = s'' \text{ and } s \neq s''' \\ \\ \text{undefined otherwise} \end{cases}$$

The component events e and e' can be visualized as vectors in state space and the resulting event as the resultant or sum of those vectors.

We stipulate next that event *e precedes* event e' if and only if e composes with e' to form a third event. In other words, we introduce a binary relation $<$ in $E'_L(x)$ as follows: Let e and e' be events occurring in a thing x with proper event space $E'_L(x)$. Then

$$e < e' =_{\text{df}} e * e' \in E'_L(x)$$

This relation $<$ is asymmetric and transitive just like $<$ and $\subset$. Hence it is a *strict partial ordering* of $E'_L(x)$. This order is not absolute (frame free) but frame dependent, because $E'_L(x)$ itself is frame dependent. Moreover the event order is not connected: There are events in $E'_L(x)$ that neither precede not succeed each other, as shown by the very definition of the composition $*$ of events. (Which is just as well, because that is exactly the case with events that are spacelike separated.)

So much for preliminaries.

5. *Interposition*

We proceed to build our first geometrical notion, namely, that of interposition or betweenness. Since we cannot avail ourselves of any ready-made geometrical concepts, we must build our notion of interposition exclusively in terms of some of the notions characterized above. We choose as definers the concepts of thing (T), basic thing (B), of being a part of a thing ($\sqsubset$), of one thing acting upon another ($\rhd$), and of the total effect of one thing on another (ϵ). All of these concepts have been elucidated in the preceding. Our definition is implicit, namely, via the following postulate:

Postulate 1. Let T be the set of things, B that of basic things, and R a ternary relation among things, Further, abbreviate "R holds between x, y, z in the given order" to $x \mid y \mid z$, where $x, y, z \in T$. Then R is the *interposition* (or *betweenness*) relation iff, for any $u, v, x, y, z \in T$,

(i) $x \mid y \mid z \Rightarrow x \neq y \neq z \neq x$ or $x = y = z$

(ii) $x \mid y \mid z \Rightarrow z \mid y \mid x$

(iii) $x \mid y \mid z \;\&\; u \mid x \mid y \;\&\; y \mid z \mid v \Rightarrow u \mid y \mid v$

(iv) $x \neq y \;\&\; x \sqsubset y \Rightarrow \neg(\exists z)(z \in T \;\&\; x \mid z \mid y)$

(v) $y \in B \Rightarrow (\exists x)(\exists z)(x, z \in B - \{y\} \;\&\; x \mid y \mid z)$

(vi) $x \mid y \mid z \;\&\; x \rhd z \Rightarrow (\exists u)\ [u \in \epsilon(x, y) \;\&\; (\forall v)(v \in \epsilon(x, z) \Rightarrow u < v)]$;

The formula $x \mid y \mid z$ is interpreted as "y interposes (or lies between) x and z." The first clause states that the interposition relation holds either among different things or, trivially, for a single thing. The second, that the relation is symmetric in the outer variables. The third, that whatever interposes between two given things interposes also between two outer things. The fourth, that nothing interposes between the part and the whole. The fifth clause asserts that any basic thing can be "surrounded" by two other basics. (Obviously this does not hold for the universe—hence the restriction of the clause to basics). The sixth clause of our axiom states that, if y interposes between x and z, and x acts on z, then some of the effects of x on y precede all of the effects of x on z. (This hypothesis can be best understood in terms of signals: If y interposes between x and z, and x acts on z, then any signal from x to z reaches y before it does z. But this reading is of course extrasystematic, for we have not characterized the notion of a signal.)

The first three clauses seem intuitive. The fourth is a sort of physical interpretation of the second component of Hilbert's second axiom group for element ary geometry (Hilbert, 1899), namely, $\neg(x \mid y \mid y)$. The fifth is a sort of dual of Hilbert's fourth component of the second axiom group—namely, that there is a thing lying between any two distinct things. The sixth clause exhibits, even more forcefully than the previous ones, the material basis of the interposition relation. Moreover it suggests that the latter could not be properly defined for a changeless universe.

6. *Separation*

There are several notions of separation. One of them is the topological notion of separation between sets (Wallace, 1941). We cannot use it because we wish to clarify the notion of separation between concrete things, not between sets, which are constructs. Other concepts of separation are metrical or quasimetrical. Any of these consists in a non-negative real-valued function d on an abstract set S obeying well known conditions. The members x and y of S are said to be separate iff $d(x,y) \neq 0$. We cannot use these notions either, because it is not our purpose to compete with physicists by assigning precise quantitative measures to the separations among things. We need a more basic, qualitative notion of separation. We shall get it with the help of the concept of interposition.

We define the separation between two things as the set of things that interpose between the given things. More precisely, we make

Definition 1. Let B be the set of basic things. The function σ: $B \times B \to 2^B$ such that

$$\sigma(x,y) = \{z \in B \mid x \mid z \mid y\} \text{ for } x,y \in B$$

is called the *basic thing separation.*

This definition, jointly with Postulate 1, entails the following consequence:

Theorem 1. For any two basic things $x, y \in B$
(i) $\sigma(x,x) = \{x\}$
(ii) $\sigma(x,y) = \sigma(y,x)$

Proof. The first part follows from clause (i) of Postulate 1. The second, from clause (ii).

Note the analogy between the set-valued function σ and the real-valued quasidistance and distance functions d mentioned at the beginning of this section. But note also the dissimilarities, such as (i) and the frame invariance of σ, which contrasts with the frame dependence of distances and quasidistances.

We now translate clause (iii) of Postulate 1:

Theorem 2. For any basic things $u, v, x, y, z \in B$, if $y \in \sigma(x, z)$, $x \in \sigma(u, y)$ and $z \in \sigma(y, v)$, then $y \in \sigma(u, v)$

Proof. By Postulate 1 (iii) and Definition 1.

We could derive further theorems but we shall not need them for our purposes. What we do need is another property of the separation function, which the previous assumptions and definitions fail to entail. We must therefore postulate it:

Postulate 2. Let x_1, x_2, y, z_1 and z_2 be basic things. If $y \in \sigma(x_1, z_1) \cap \sigma(x_2, z_2)$ and $x_1 \neq z_1, x_2 \neq z_2$, then there are basic things x_3 and z_3 such that $x_3 \neq z_3$ and $y \in \sigma(x_3, z_3) \subset \sigma(x_1, z_1) \cap \sigma(x_2, z_2)$

We now have all we need to build the notion of thing space:

Definition 2. The set B of basic things, together with the separation function σ, is called the *basic thing space.* Symbol: $\vartheta = \langle B, \sigma \rangle$.

In other words, what we call the "thing space" is nothing but *the collection* *of spaced things*, or the collection of things related by their mutual separations. This is all the philosopher needs, for ϑ has been built in terms of a few extremely general (ontological or protophysical) concepts. We may call ϑ a *philosopher's space.*

7. *Basic Properties of Physical Space*

The thing space ϑ introduced by Definition 2 may satisfy the philosopher, but not the mathematician or the physicist: It has too poor a structure to qualify as physical space. The least we must do is to topologize either the set B of basic things or its power set 2^B. Fortunately the separation function σ will do the trick. Indeed we have

Theorem 3. The family of sets of basic things

$\tau = \{X \in 2^B \mid \text{for all } y \in X, \text{ there are } x, z \text{ in } B - \{y\} \text{ such that } y \in \sigma(x,z) \subset X\}$

is a topology for B.

Proof. By Postulate 1(v), $B \in \tau$. Besides, it is obvious that $\phi \in \tau$ and that every union of members of τ belongs to τ. Finally, by Postulate 2, the intersection of any two elements of τ is also in τ. Hence τ is a topology for B.

We shall assume that the set B of basic things, equipped with the topology τ, is equal to physical space:

Postulate 3. The topological space $\Sigma = \langle B, \tau \rangle$ is equal to the physical (real, ordinary) space.

However, we have so far little justification for assuming this hypothesis. For one thing we have assigned Σ no definite dimensionality, whereas we feel certain that physical space is three dimensional. For another we want Σ to be connected. These and other shortcomings will be remedied by foisting certain properties on Σ. That is, we shall force Postulate 3 to become true.

To begin with, let us define, in the usual way, closures in Σ. If $A \in \tau$, then

$$\text{Cl}\, A =_{\text{df}} \{x \in B \mid \text{for all } V \text{ in } \tau \text{ containing } x,\ V \cap A \neq \phi\}.$$

For these closures to be of any help, they must be large enough. This is ensured by the following postulate:

Postulate 4. Every pair of basics is in the τ-closure of their separation. I.e.,

$$\text{for any } x, y \in B, \text{ if } \sigma(x,y) \neq \varnothing, \text{ then } x, y \in \text{Cl}\, \sigma(x,y)$$

By taking $x = y$, it follows immediately that every finite subset of B is closed. In other words, we have the following:

Corollary 1. The physical space $\Sigma = \langle B, \tau \rangle$ is a T_1-space.

We need one more assumption to convert Σ into a Hausdorff (or T_2) space. This step will be taken in the following section.

8. *Further Properties of Physical Space*

Let us recall that our goal is to endow Σ with the minimal set of properties demanded by contemporary physical theory. More precisely we wish to render Σ a connected and metrizable space locally homeomorphic to $\mathbb{R}^3$. More briefly, we want Σ to be a three-dimensional connected manifold, hence a locally Euclidean space.

Now, it has been known for a long time that it is possible to express every theorem in Euclidean three-dimensional geometry with the sole help of the concepts of sphere and of inclusion. In fact these are the sole extralogical undefined notions in the set of postulates for elementary geometry proposed by Huntington (1913). We shall adopt these postulates but, instead of taking the notions of sphere and of inclusion as primitive, we shall define them in terms of some of our previous concepts. This will endow Huntington's postulates with a physical (or rather protophysical) interpretation and so will authorize us to claim that they characterize physical space (in the small).

Our first task then is to define the concept of a sphere. We will define a sphere lying between two given basic things as the closure of the separation between those things. More precisely, we make the following definition:

> *Definition* 3. For any pair $x, z \in B$ of distinct basics, the *spheres* lying between x and z are
>
> $$Sxz = \{\mathrm{Cl}\,\sigma(u, v) \mid u, v \in B \ \&\ \varnothing \neq \mathrm{Cl}\,\sigma(u, v) \subset \sigma(x, z)\}$$

The minimal spheres are the points—which need not be "unextended." The set of minimal spheres will prove to be mappable on $\mathbb{R}^3$.

We now assume that our spheres satisfy Huntington's postulates:

> *Postulate* 5. For any given pair $x, z \in B$ of distinct basics, the structure $\langle Sxz, \subset\rangle$, where $Sxz \neq \varnothing$, satisfies (is a model of) the Huntington (1913) postulates.

Hence we have the following:

> *Theorem* 4. The nonempty separation $\sigma(x, z)$ between two distinct basic things $x, z \in B$ is homeomorphic to three-dimensional Euclidean space.

Proof. The minimal spheres are the single points of $\sigma(x, z)$. By Huntington's (1913) Theorem 47, this collection can be topologized in such a way that it is homeomorphic to $\mathbb{R}^3$. Now, a basis for this latter topology consists of all the τ-interiors of elements of Sxz. Hence this topology is identical with the relative topology of $\sigma(x, z)$.

We are now justified in framing the following:

> *Definition* 4. Every member $\mathrm{Cl}\,\sigma(u, v)$ of the family of spheres Sxz lying between the things x and z is said to be a *Euclidean ball.*

Remember now that physical space is a T_1 space (Corollary 1). Since Postulate 5 supplies regularity, we obtain the following:

> *Corollary* 2. $\Sigma = \langle B, \tau\rangle$ is a regular three-manifold without boundary.

We have come near the goal. We shall attain it by adding one last assumption, namely, that any two basic things can be bridged by a string of spheres. More precisely, we assume the following:

> *Postulate* 6. The set B of basics contains two sequences $x_1, x_2, \ldots$ and $z_1, z_2, \ldots$ such that $x_i \neq z_i$ for each i and, for every pair a, b in B, there is a simple chain
>
> $$a \in C_1, C_2, \ldots, C_n \ni b$$
>
> such that every C_i is of the form $\sigma(x_j, z_j)$.

Finally we can prove what we wanted:

> *Theorem* 5. $\Sigma = \langle B, \tau \rangle$ is a connected, second countable and metrizable three-manifold without boundary.

Proof. Being homeomorphic to $\mathbb{R}^3$, every $\sigma(x_i, z_i)$ is connected and Lindelöf. Hence, by Postulate 6, $\langle B, \tau \rangle$ is connected and Lindelöf. Now, every regular Lindelöf space is paracompact and every paracompact locally metrizable space is metrizable. Therefore $\langle B, \tau \rangle$ is connected, Lindelöf, and metrizable. Finally, $\langle B, \tau \rangle$ is second countable because, in metrizable spaces, the properties being Lindelöf and of being second countable are equivalent.

In other words, *physical space is a three-dimensional connected manifold.* And, because Σ is metrizable, the physicist may assign it an adequate (compatible) metric. We shall abstain from doing so because our goal was to build a concept of physical space broad enough that it might be used in the axiomatic foundations of any of the current physical theories. Besides, the theories of relativity have taught us that any attempt to assign space a metric, independently of time, is bound to fail: Only space-time can be assigned the proper metric.

9. *Comparison with Physical Geometries*

Why should we require Σ to have precisely the properties postulated or deduced in the previous sections? Why could we not have abstained from assuming some of them? Why could we not hypothesize an entirely different set of properties for physical space? The answer to these questions depends upon the relation of one's philosophy with science, hence indirectly with reality.

In a totally a priori philosophy, physical space can be assigned any structure whatsoever. For example, one could postulate—as Whitehead (1919) and Lucas (1973) have done in the wake of Kant—that physical space is globally Euclidean. On the other hand no such freedom exists in a philosophy that seeks to be continuous with science. In a science-oriented philosophy one requires the geometry of ordinary space to agree with physics. Moreover one lets physics have the upper hand.

Now, current physics happens to assume that physical space is a three-dimensional differentiable manifold. (See, e.g., Trautman, 1965). This is necessary to write down the basic equations of contemporary physics—though usually insufficient to solve them. (We disregard here the speculative deviant theories.

If any of them were shown to be true then we would have to change our geometry.) Surely most theories require some additional structure, if not at the foundational level at least at the time of solving problems. For example, classical mechanics and nonrelativistic quantum mechanics require, in their usual formulations, that space be globally Euclidean; classical electrodynamics requires space to be affine (van Dantzig, 1935), and the relativistic theory of gravitation assumes that space, or rather space-time, is Riemannian. But the assumption common to all of these specifications is that physical space is a three-dimensional connected manifold. This being the minimal geometrical assumption, it should be a necessary and sufficient constraint for philosophy —until further notice.

The geometry described in the previous sections is then compatible with the mainstream of contemporary physics. A new physics might call for a new protophysical geometry. Two radical changes of the sort have been suggested several times. One is that the spatial generalized continuum may have to be replaced by a discontinuous or atomic space with a fundamental or minimal length built into it. Another, that space-time may be fluctuating, our current metrics being sorts of statistical averages. However, none of these ideas seems to have been carried beyond the heuristic stage. The fact is that the theories actually employed by physicists make no assumptions about physical space contradicting our results.

In sum, we may declare our protogeometry true because contemporary physics says so. But at the same time we should be prepared to see it revised or even revolutionized by new developments in physics.

10. *Concluding Remarks*

We have obtained a solution to the problem we set out to solve, namely, that of building a theory of physical space based on the concept of a changing thing. (This does not mean that changing things are more fundamental than space: Only the corresponding categories are ordered.) Ours is then a relational theory of physical space.

However, ours is not the only possible theory of the kind. For example there is Basri's (1966) rigorous relational theory. But, because it admits only particles as basics, this theory violates the spirit of the two relativities, which have the field concept at heart. Besides, Basri's is a subjectivist theory because it is concerned with human observers and their sensations and operations: As such it is observer-bound rather than observer-free as all scientific theories are supposed to be. (For the latter point see Bunge, 1973.)

Besides there is Penrose's (1971) rigorous and moreover objectivist relational theory of space. But, because it is based on the concept of angular momentum, it does not seem to be general enough to serve as a foundation for all physical theories. In fact there are things with no intrinsic angular momentum. Our own theory is, on the other hand, based on extremely general ideas, hence less sensitive to scientific change.

Our theory has one seemingly unsatisfactory feature, namely that it postulates that physical space is three dimensional instead of explaining this fact. (This hypothesis is contained in Postulate 5.) This assumption may seem arbitrary. It is not, for it is suggested by the actual behavior of real things: If things behaved differently, then physical space might not be three dimensional. In other words, the three dimensionality of space is rooted in the known physical laws. (See, e.g., Penney, 1965.) What would the dimensionality of physical space be if things possessed ("obeyed") laws different from those we know? This question can receive a partial answer, namely, along the following lines. Take for instance the pervasive processes of wave propagation and write out any wave equation in spherical coordinates in an n-dimensional manifold. Any such equation will contain a radial term with a factor $(n - 1)$. So, if experiment were to show that wave propagations proceed according to, say, $n = 2$ or $n = 4$, we would have to conclude that physical space is two-dimensional or four-dimensional, respectively. But of course this is not the case. In short, experiment points, in however devious a fashion, to the three-dimensionality of ordinary space. (Note by the way that, on a nonrelational theory of space, one would ask instead the question: What would things look like if space were not three-dimensional? But this question cannot be answered except dogmatically.) Hence in postulating that ordinary space is three-dimensional we just bow to experience.

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References

Basri, S. A. (1966). *A Deductive Theory of Space and Time*, Amsterdam.
Bunge, M. (1973). *Philosophy of Physics.* Dordrecht.
Bunge, M. (1974). *International Journal of General Systems*, **1**, 229.
Bunge, M. (1977). In *Systems: Approaches, Theories, Applications*, Hartnett, W. E., ed., Dordrecht and Boston.
Bunge, M. and Sangalli, A. (1977). *International Journal of General Systems*, **3**, 146.
Hilbert, D. (1899). *Grunlagen der Geometrie*, 8th edition, Stuttgart, 1956.
Huntington, E. V. (1913). *Mathematische Annalen*, **73**, 522.
Lucas, J. R. (1973). *A Treatise on Time and Space.* London.
Penney, R. (1965). *J. Mathematical Physics* **6**, 1607.
Penrose, R. (1971). In *Quantum Theory and Beyond*, Bastin, T., ed., Cambridge.
van Dantzig, D. (1935). *Proceedings of the Cambridge Philosophical Society*, **30**, 421.
Wallace, A. D. (1941). *Annals of Mathematics* (2), **42**, 687.
Whitehead, A. N. (1919). *An Enquiry Concerning the Principles of Natural Philosophy*, Cambridge.